Effects of Voids on the Response of a Rubber Poker Chip Sample. III

P. A. KAKAVAS* and P. J. BLATZ

Department of Chemical Engineering, University of Southern California, Los Angeles, California 90089

SYNOPSIS

The effects of voids on the response of a rubber poker chip sample are examined. A theoretical estimation of the diametral contraction of the sample was performed, using the linear theory of stress analysis. Experimental measurements of the lateral contraction at the middle plane of the poker chip elastomer specimen have shown that the testing rubber is not incompressible. By comparing the experimental data with the theoretical predicted equation, the value of the Poisson's ratio ν_{eff} was found to be 0.487, for a given aspect ratio a^* of the sample. A theoretical equation for the volume dilatation of the poker chip rubber sample was developed. Using the given aspect ratio, the value of ν_{eff}, and the experimental stress/strain curve of the sample, an estimation of the volume dilatation was formed. The effective Poisson's ratio was also found using the linear stress analysis, by comparing the developed mathematical equations for an incompressible rubber with voids with a compressible one.

INTRODUCTION

In the previous articles,[1,2] it was experimentally shown (using the acoustic emission technique) that microvoids exist within a poker chip sample whenever it is subjected either to tension or compression. It was found that the voids are responsible for the drop of the apparent modulus of the elastomeric poker chip specimen.

The aim of this article is to propose a theoretical background for the decrease of the apparent modulus. First of all, a mathematical equation was developed for the diametral contraction of the poker chip sample at the middle plane, using the *linear* stress analysis.

Second, experimental measurements were made on the deflected sample in order to measure the lateral contraction in the middle plane as a function of the strain. Experimental measurements have shown that the deflected rubber poker chip is *no* longer incompressible due to microvoids.

A mathematical expression was developed for the volume dilatation of the sample, as will be presented in Section 5. Using Schapery's equation[3-6] for the normalized apparent modulus M/E and the experimental data, an estimation of the effective Poisson's ratio ν_{eff} was made.

EXPERIMENTAL

Theoretical Prediction of the Diametral Contraction of a Poker Chip Elastomer Sample

The geometry and the coordinate system of the poker chip sample used for the analysis are shown in Figure 1. Let a and h denote the radius and the thickness of the specimen, respectively. Following the *linear* stress analysis, it can be shown (see Appendix) that the displacement $u(r, z)$ and $w(z)$ are given by

$$u(r, z) = AI_1(\rho)\left(1 - 4\frac{z^2}{h^2}\right) \tag{1a}$$

* To whom correspondence should be addressed at 135 Korinthou Street, 262 23 Patras, Greece.

Journal of Applied Polymer Science, Vol. 43, 1081–1086 (1991)
 CCC 0021-8995/91/061081-06$04.00

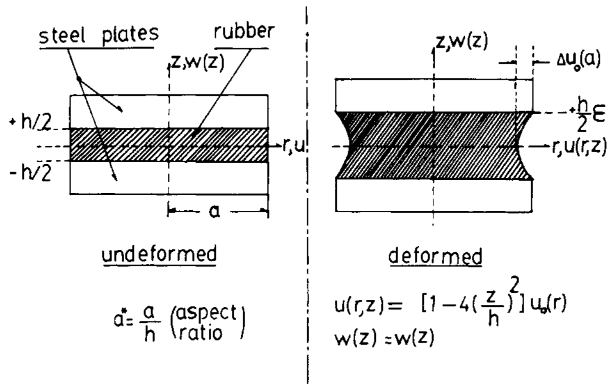

Figure 1 Triaxial poker chip test and the coordinate system used in the stress analysis.

$$w(z) = w(z) \tag{1b}$$

where

$$\rho = r\chi; \quad \chi = \frac{2}{h}\sqrt{\frac{3 - 6\nu}{2 - 2\nu}}$$

and $I_1(\rho)$ and ν denote the modified Bessel function of *first* order and the Poisson's ratio, respectively. Coefficient A is given by

$$A = -\frac{3\nu a \cdot \epsilon}{2\{(1-\nu)\chi a I_0(\chi a - (1-2\nu)I_1(\chi a)\}} \tag{1c}$$

where $I_0(\chi a)$ and ϵ defines the modified Bessel function of *zero* order and the strain in the z direction.

Substitution of eq. (1c) into eq. (1a) yields the *lateral contraction* $-u_0(a)/a$ at the middle plane of the poker chip and the outer edge ($z = 0$, $r = a$), i.e.,

$$-\frac{u_0(a)}{a} = \frac{3\nu \cdot \epsilon}{2[(1-\nu)m - (1-2\nu)]} \tag{2a}$$

where parameter m is given by

$$m = \chi a \frac{I_0(\chi a)}{I_1(\chi a)} \tag{2b}$$

When $a \rightarrow 0$, then $\chi a \rightarrow 0$, and by expanding the $I_0(\chi a)$ and $I_1(\chi a)$ around zero, it can be easily shown that $m \rightarrow 2$ and eq. (2a) yields

$$-\frac{u_0(a)}{a} \rightarrow \frac{3\nu}{2} \cdot \epsilon \tag{3a}$$

When $a \rightarrow \infty$, then $I_0(\chi a)/I_1(\chi a) \approx 1$ and $m \rightarrow \chi a$, hence,

$$-\frac{u_0(a)}{a} = \frac{3\nu \cdot \epsilon}{2[a^*\sqrt{6(1-2\nu)(1-\nu)} - (1-2\nu)]} \tag{3b}$$

where $a^* = a/h$ (aspect ratio).

A typical value of a^* for our experimental studies[1,2] is about $a^* = 8$. For this value of the aspect ratio, parameter m is close to χa; hence, the *lateral contraction* $-u_0(a)/a$ at the central plane is given via eq. (3b). For a more precise value of the lateral contraction, we use eq. (2a), estimating the modified Bessel functions by using either tables or IMSL computer subroutines. A plot of the *diametral contraction* $\delta = -u_0(a)/a\epsilon$ as a function of the effective Poisson's ratio ν_{eff} is given in Figure 2.

Experimental Measurements of the Diametral Contraction of an Elastomer Poker Chip Sample

The experimentally determined lateral contraction, $-u_0(a)/a$, at the middle plane of the poker chip sample subjected to tension and compression as a function of strain is given in Figure 3. It was experimentally measured with a caliper. All the data points fall on a straight line whose slope is $\gamma = \mathbf{0.269}$. The mathematical equation that describes the experimental relationship between the lateral contraction and strain ($\Delta L/L_0$) is given by

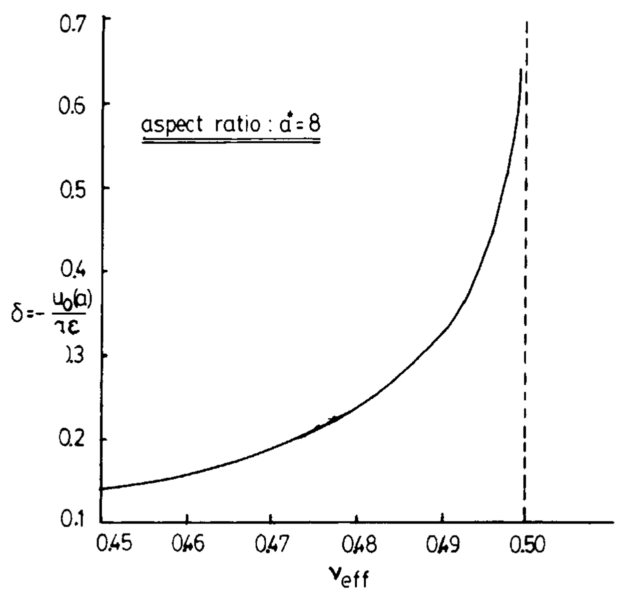

Figure 2 Diametral contraction δ vs. the effective Poisson's ratio ν_{eff}.

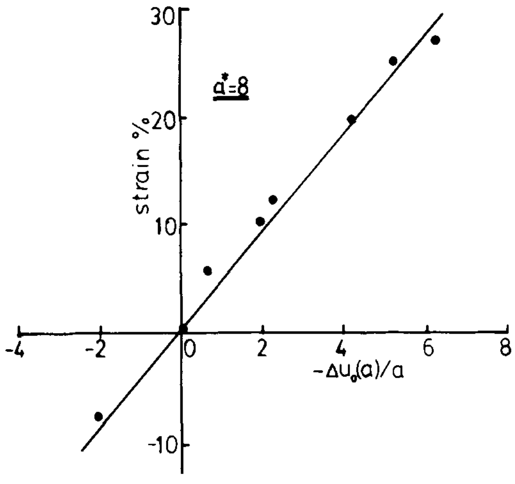

Figure 3 Observed values of the lateral contraction $-\Delta u_0(a)/a$ as a function of the strain ϵ of the rubber poker chip sample.

$$-\frac{u_0(a)}{a} = 0.269 \cdot \epsilon \qquad (4)$$

where a is the radius of the elastomer disc. Equation (4) indicates that the diametral contraction derived by the longitudinal strain, $\delta = -u_0(a)/a \cdot \epsilon$, is equal to $\gamma = 0.269$. Referring to Figure 2, the effective Poisson's ratio is equal to $\nu_{\text{eff}} = \mathbf{0.487}$. This low value of the Poisson's ratio indicates that the elastomer poker chip sample (whose chemical composition was given in ref. 1) is *not* incompressible.

However, we know that a homogeneous unfilled nitrile rubber is nearly *incompressible* (i.e., $\nu = 0.5$). Why does such an apparent contradiction exist? We attribute this contradiction to the existence of *voids* in the poker chip test sample. It was shown in previous papers[1,2] that the voids are created during the molding process of the sample. The magnitude of the voids within the sample subjected to tension and compression was estimated using the frequency spectrum of the detected acoustic waveforms.[1,2]

Volume Dilatation, $\Delta V/V_0$

It can be shown (see Appendix) that the volume dilatation of the elastomer poker chip specimen is given by

$$\frac{\Delta V}{V_0} = \left[\frac{(1-\nu)m - 1}{(1-\nu)m - (1-2\nu)}\right] \cdot \epsilon \qquad (5)$$

where $V_0 = \pi a^2 h$, the initial volume of the sample; $\Delta V = V - V_0$ (V = final volume); ν denotes the effective Poisson's ratio; m is given via eq. (2b); and ϵ defines the longitudinal strain. When $a \rightarrow 0$, then $m \rightarrow 2$ and eq. (5) yields

$$\frac{\Delta V}{V_0} = (1 - 2\nu) \cdot \epsilon \qquad (6a)$$

For an incompressible elastomer disc, $\nu \rightarrow 0.5$; hence, $\Delta V/V_0 \rightarrow 0$, i.e., no volume dilatation exists. When $a \rightarrow \infty$, then $m \rightarrow \chi a$ and eq. (5) yields

$$\frac{\Delta V}{V_0} = \left[\frac{a^*\sqrt{6(1-2\nu)(1-\nu)} - 1}{a^*\sqrt{6(1-2\nu)(1-\nu)} - (1-2\nu)}\right] \cdot \epsilon \qquad (6b)$$

where a^* is the aspect ratio ($a^* = a/h$), which is about $a^* = 8$ for our experimental studies.[1,2] The parameter χ was given in Section 3.

Substituting the values of $a^* = 8$ and $\nu = 0.487$ into eq. (6b), the volume dilatation of the tested elastomer disc is equal to

$$\frac{\Delta V}{V_0} = 0.597\epsilon \qquad (7)$$

Hence, the volume dilatation for the rubber poker chip considered here as a function of the strain (at low strain) is a *linear* function of strain whose slope is equal to $\beta = \mathbf{0.597}$. A typical stress–strain curve at low strain of a poker chip sample is given in Figure 4. We observed that the yield point occurs at 10% strain, where the volume dilatation according to eq. (7) is **0.0597.**

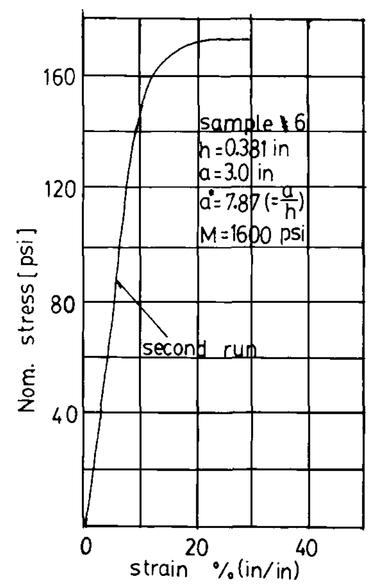

Figure 4 Applied stress on the poker chip specimen as a function of strain (at low strain).

Determination of the Apparent Modulus, M, of an Elastomer Poker Chip

It can be shown (see Appendix) that the normalized apparent modulus, M/E, of a poker chip specimen is given by

$$\frac{M}{E} = \frac{1}{(1-2\nu)(1+\nu)} \times \left[(1-\nu) - \frac{2\nu^2}{(1-\nu)m - (1-2\nu)} \right] \quad (8)$$

where E defines the Young's modulus of the material. For the elastomeric material used in this study,[1,2] we determined E to be equal[1,2] to 1.413 N/mm^2.

When $a \to 0$, then $m \to 2$ and eq. (8) yields $M/E \to 1$. When $a \to \infty$, then $m \to \chi a$ and eq. (8) yields

$$\frac{M}{E} = \frac{1}{(1-2\nu)(1+\nu)} \left[(1-\nu) - \frac{2\nu^2}{a^* \sqrt{6(1-2\nu)(1-\nu)} - (1-2\nu)} \right] \quad (9)$$

Equation (9) yields the ratio M/E as a function of the aspect ratio a^* and the Poisson's ratio ν. Letting $a^* = 8$, then using either the tables or the IMSL computer subroutines, the ratio M/E can be determined for different values of ν. Figure 5 shows the normalized modulus, M/E, versus the Poisson's ratio ν.

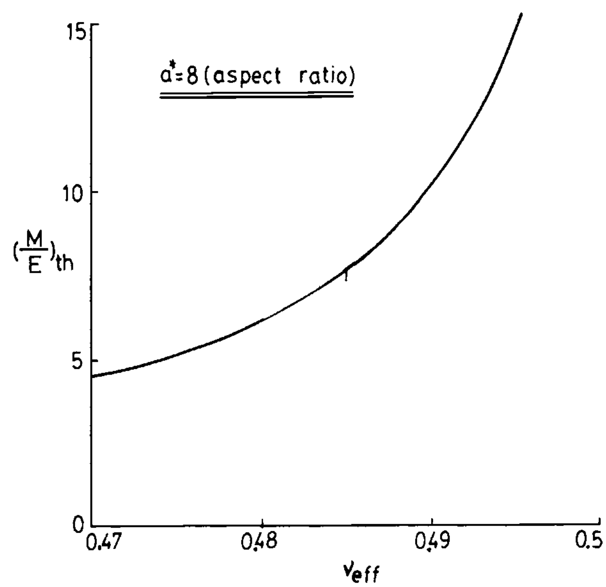

Figure 5 The theoretical predicted normalized modulus, $(M/E)_{th}$, of the poker chip as a function of the effective Poisson's ratio ν_{eff} ($a^* = 8$).

Our experimental work has shown that the ratio M/E is equal to 7.87, which leads to a value of the Poisson's ratio approximately equal to ν = **0.487**. Therefore, the effective Poisson's ratios determined from the stress–strain curve and from the diametral contraction have the same value.

CONCLUSIONS

In this study, we have developed a mathematical background for the growth of microvoids within a deflected elastomer poker chip disc. Using *linear* stress analysis, we developed a mathematical expression for the diametral contraction of the sample as a function of the Poisson's ratio, the aspect ratio of the sample and the strain. Experimental measurements of the diametral contraction as a function of the strain have shown that all the points fall on a straight line of slope γ = 0.269 (for small strain). Using the experimental data and the theoretically predicted equation, the effective Poisson's ratio was found to be equal to ν_{eff} = **0.487**. Hence, the rubber subjected either to compression or tension is no longer incompressible. The same result can be arrived at by measuring the volume dilatation of the poker chip sample. The effective Poisson's ratio ν_{eff} can also be determined using the developed mathematical equation for the normalized apparent modulus, M/E, of the rubber disc.

APPENDIX

Computation of the Displacement and the Stress Field within a Poker Chip Specimen with Voids

Let us assume that the displacement field within the deformed poker chip sample is given by

$$u(r,z) = u_0(r)(1 - 4z^2/h^2) \quad (A.1)$$

$$w(z) = w(z)(\text{arbitrary}) \quad (A.2)$$

In eq. (1) it was assumed that the profile in the z-direction is parabolic. Since,

$$u(r, h/2) = 0 \quad (A.3)$$

the incompressibility equation (written in cylindrical coordinates) implies that

$$w_z(z = h/2) = 0 \tag{A.4}$$

where w_z denotes the derivative of w with respect to the z-coordinate. Also, the w-displacement on the upper plate of the sample is given by

$$w(h/2) = \epsilon h/2 \tag{A.5}$$

where ϵ denotes the strain within the poker chip sample.

Substitution of the displacement components u, w into *linear* stress-displacement *axisymmetric* equations,[7] we obtain

$$\frac{1-2\nu}{2G}\sigma_r = \left[(1-\nu)u_0' + \nu\frac{u_0}{r}\right]\left(1 - \frac{4z^2}{h^2}\right) + \nu w' \tag{A.6.1}$$

$$\frac{1-2\nu}{2G}\sigma_\theta = \left[\nu u_0' + (1-\nu)\frac{u_0}{r}\right]\left(1 - \frac{4z^2}{h^2}\right) + \nu w' \tag{A.6.2}$$

$$\frac{1-2\nu}{2G}\sigma_z\left[\nu(u_0') + \frac{u_0}{r}\right]\left(1 - \frac{4z^2}{h^2}\right) + (1-\nu)w' \tag{A.6.3}$$

$$\frac{1-2\nu}{2G}\tau_{rz} = \frac{1-2\nu}{2}\left[-\frac{8u_0 z}{h^2}\right] \tag{A.6.4}$$

where G and ν represent the shear modulus and the Poisson's ratio, respectively.

Taking the average of eqs. (6) along the z-direction, we get

$$\frac{1-2\nu}{2G}\langle\sigma_r\rangle = \frac{2}{3}\left[(1-\nu)u_0' + \nu\frac{u_0}{r}\right] + \nu\epsilon \tag{A.7.1}$$

$$\frac{1-2\nu}{2G}\langle\sigma_\theta\rangle = \frac{2}{3}\left[\nu u_0' + (1-\nu)\frac{u_0}{r}\right] + \nu\epsilon \tag{A.7.2}$$

$$\frac{1-2\nu}{2G}\langle\sigma_z\rangle = \frac{2}{3}\left[u_0' + \frac{u_0}{r}\right] + (1-\nu)\epsilon \tag{A.7.3}$$

$$\langle\tau_{rz}\rangle = 0 \tag{A.7.4}$$

$$\frac{V}{G} = -\frac{4u_0}{h} \tag{A.7.5}$$

where V is the shear at $z = h/2$, and

$$\langle * \rangle = \frac{1}{h}\int_{-h/2}^{h/2} (*)\,dz \tag{A.7.6}$$

The *equilibrium equations* are[7,8]

$$\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial\tau_{rz}}{\partial z} = 0 \tag{A.8.1}$$

$$\frac{\partial\tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + \frac{\partial\sigma_z}{\partial z} = 0 \tag{A.8.2}$$

The *average* of eq. (8.1) upon z is

$$\langle\sigma_r\rangle + \frac{\langle\sigma_r\rangle - \langle\sigma_\theta\rangle}{r} + \frac{2V}{h} = 0 \tag{A.9}$$

Substitution of eqs. (7) into eq. (9) yields the following *modified Bessel equation*:

$$u_0'' + \frac{u_0'}{r} - \left(\chi^2 + \frac{1}{r^2}\right)u_0 = 0 \tag{A.10}$$

where

$$\chi = \frac{2}{h}\left[\frac{3-6\nu}{2-2\nu}\right]^{1/2} \tag{A.11}$$

The solution of eq. (10) is given as a function of the *modified Bessel functions* $I_1(\rho)$ and $K_1(\rho)$:

$$u_0 = AI_1(\rho) + BK_1(\rho) \tag{A.12}$$

Since $u_0(\rho = 0)$ is finite and $K_1(0)$ is infinite, the constant B must vanish. Hence, eq. (1) takes the form

$$u(\rho, z) = AI_1(\rho)\left[1 - \frac{4z^2}{h^2}\right] \tag{A.13}$$

At the free surface of the poker chip, the normal and the shear components of the stresses must vanish, i.e.,

$$\sigma_r(\rho = \chi a) = \tau_{rz}(\rho = \chi a) = 0 \tag{A.14}$$

Substitution of eq. (13) into eq. (6.1) and using eq. (14), one gets

$$A = \frac{-1.5\nu\epsilon a}{(1-\nu)\chi a I_0(\chi a) - (1-2\nu)I_1(\chi a)} \tag{A.15}$$

The dilatation θ is given via the following equation:

$$\theta = u_r + \frac{u}{r} + w' \tag{A.16}$$

where $u_r \equiv \partial u/\partial r$ and the average upon z is

$$\langle\theta\rangle = \frac{2}{3}\left[u_0' + \frac{u_0}{r}\right] + \epsilon \tag{A.17}$$

Replacing u_0 and its derivative u_0' into eq. (17), we obtain

$$\langle\theta\rangle = \epsilon\frac{(1-2\nu)[\chi a I_0(\chi a) - I_1(\chi a)]}{(1-\nu)\chi a I_0(\chi a) - (1-2\nu)I_1(\chi a)} \tag{A.18}$$

The volume fraction of the voids is

$$\frac{\Delta V}{V_0} = \int_0^a \frac{2\pi r\langle\theta\rangle}{\pi a^2}\,dr \tag{A.19}$$

After integration, the $\Delta V/V_0$ becomes

$$\frac{\Delta V}{V_0} = \epsilon\left[\frac{(1-\nu)m - 1}{(1-\nu)m - (1-2\nu)}\right] \quad (A.20)$$

where

$$m = \chi a \frac{I_0(\chi a)}{I_1(\chi a)} \quad (A.21)$$

With substitution of u_0 into eq. (7.3) and integration upon r, we obtain the load T on the poker chip plates, i.e.,

$$T = \frac{E\epsilon}{(1-2\nu)(1-\nu)} \times \left[(1-\nu) - \frac{2\nu^2}{(1-\nu)m - (1-2\nu)}\right] \quad (A.22)$$

Parameter m is defined via

$$m = \chi a \frac{I_0(\chi a)}{I_1(\chi a)} \quad (A.23)$$

T/ϵ defines the modulus of the poker chip sample; hence,

$$\frac{M}{E} = \frac{1}{(1-2\nu)(1-\nu)} \times \left[(1-\nu) - \frac{2\nu^2}{(1-\nu)\,m - (1-2\nu)}\right] \quad (A.24)$$

REFERENCES

1. P. Kakavas and W. V. Chang, *J. Appl. Polym. Sci.*, to appear.
2. P. Kakavas and W. V. Chang, *J. Appl. Polym. Sci.*, to appear.
3. C. H. Lindsey, R. A. Schapery, M. Williams, and A. Zak, Aerospace Research Laboratories Report ARh., 63-152, September, 1963.
4. G. H. Lindsey, *J. Appl. Phys.*, **38**(12), 4843 (1967).
5. M. Williams and R. A. Schapery, *Int. J. Fracture Mech.*, **1**, 64 (1965).
6. G. Lindsey, R. Schapery, et al., GALTIC SM 63-6, 1963.
7. I. S. Sokolnikof, *Mathematical Theory of Elasticity*, 2nd Edition, McGraw-Hill, New York, 1956.
8. S. Timoshenko and J. N. Goodier, *The Theory of Elasticity*, 2nd Edition, McGraw-Hill, New York, 1951.

Received October 18, 1990
Accepted December 20, 1990